1 A SQUARE AND A CUBE

Numbers reveal beautiful patterns. Squares and cubes are two such special patterns.

IN THIS CHAPTER YOU WILL LEARN

- Square numbers and their properties
- Square roots
- Cube numbers and their properties
- Cube roots
- Patterns, estimates and number sense
- Historical connections and puzzles

THE LOCKER PUZZLE (Chapter Opening)

100 lockers numbered 1 to 100 are toggled in rounds.

- Person 1 opens every locker.
- Person 2 toggles every 2nd locker.
- Person 3 toggles every 3rd locker … and so on.

1	2	…	10
11	12	…	20
…	…	4	…
91	92	…	100

A locker is toggled as many times as the number of factors of its number. Only lockers with an odd number of factors remain open — i.e., the square numbers.

Open lockers: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100.
The passcode was hidden in the lockers touched exactly twice — the prime numbers: 2 – 3 – 5 – 7 – 11.

KEY DEFINITIONS

- **Square Number:** A number obtained by multiplying a number by itself.
  $n \times n = n^2$ (Read as 'n squared')
- **Perfect Square:** The squares of natural numbers.
- **Square Root:** If $y = x^2$, then x is the square root of y.
  Denoted by $\sqrt{y}$ (positive square root).
  Every perfect square has two integral square roots: $+\sqrt{y}$ and $-\sqrt{y}$.
- **Cube (or Cube Number):** A number obtained by multiplying a number by itself three times.
  $n \times n \times n = n^3$ (Read as 'n cubed')
- **Perfect Cube:** The cubes of natural numbers.
- **Cube Root:** If $y = x^3$, then x is the cube root of y.
  Denoted by $\sqrt[3]{y}$.

SQUARE NUMBERS

Squares of first 10 natural numbers

n	n^2	Value	n	n^2	Value
1	1^2	1	6	6^2	36
2	2^2	4	7	7^2	49
3	3^2	9	8	8^2	64
4	4^2	16	9	9^2	81
5	5^2	25	10	10^2	100

Properties & Patterns

- Perfect squares end with 0, 1, 4, 5, 6 or 9.
- Squares can have only an even number of zeros at the end.
- A number ends in 2, 3, 7 or 8 → Not a square.
- If a number has 1 or 9 in the units place, its square ends in 1.
- If a number has 6 in the units place, its square also ends in 6.
- Squares increase by consecutive odd numbers.

SQUARES AS SUM OF ODD NUMBERS

$1 = 1$
$4 = 1 + 3$
$9 = 1 + 3 + 5$
$16 = 1 + 3 + 5 + 7$
$25 = 1 + 3 + 5 + 7 + 9$
$36 = 1 + 3 + 5 + 7 + 9 + 11$

In general: n^2 = sum of first n odd numbers.

FACTORS AND SQUARES

- A locker is toggled as many times as the number of factors of its number.
- Numbers with an odd number of factors are exactly the square numbers.
- Every factor (except in a square number) has a partner factor. In a square number, one factor pairs with itself.

6 : 1×6, 2×3	4 : 1×4, 2×2	9 : 1×9, 3×3	1 : 1×1
Factors: 1, 2, 3, 6	Factors: 1, 2, 4	Factors: 1, 3, 9	Factors: 1
(Even)	(Odd)	(Odd)	(Odd)
Closed	Open	Open	Open

Only square numbers remain open.
Hence, open lockers are the squares.

FINDING SQUARE ROOTS

Methods

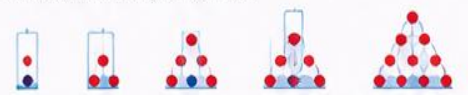
1. List squares in order.
2. Use: n^2 = sum of first n odd numbers. Subtract successive odd numbers.
3. Prime factorisation: factors in pairs.
4. Estimation:
   - Find nearest perfect squares.
   - Check units digit (√ ends in 0,1,2,3,4,5,6,7,8,9).

Examples

- $\sqrt{1936} = 44$ (since $44^2 = 1936$)
- $\sqrt{576} = 24$ (since $576 = (2^2)(3^2)(4^2) = (2\times3\times4)^2$)
- $40 < \sqrt{1936} < 50$, last digit 4 or 6
  $45^2 = 2025 > 1936 \Rightarrow \sqrt{1936} = 44$

MORE ABOUT SQUARES

- The difference between consecutive squares is an odd number.
- Numbers between n^2 and $(n+1)^2$ are:
  $n^2+1, n^2+2, \ldots, (n+1)^2-1$ → Total $2n$ numbers.
- There are n perfect squares between $(n-1)^2$ and n^2.
  e.g., between 0 – 100: 1 – 10 squares, 100 – 200: 11 – 14 squares, 200 – 300: 15 – 17 squares, …
- Relation with triangular numbers:



1, $1 = 1^2$ | $1+3=4 = 2^2$ | $1+3+5=9 = 3^2$ | $1+3+5+7=16 = 4^2$ | $1+3+5+7+9=25 = 5^2$

CUBE NUMBERS

Cubes of first 10 natural numbers

n	n^3	Value	n	n^3	Value
1	1^3	1	6	6^3	216
2	2^3	8	7	7^3	343
3	3^3	27	8	8^3	512
4	4^3	64	9	9^3	729
5	5^3	125	10	10^3	1000

Properties & Patterns

- Possible last digits: 0, 1, 8, 9, or even number of zeros.
- A cube cannot end with exactly two zeros.
- Number of digits in cubes:
  1–9 (1 digit), 10–21 (2 digits), 22–46 (3 digits), …
- n^3 increases by adding $(3n^2 + 3n + 1)$ each time.

CUBES AS SUM OF CONSECUTIVE ODD NUMBERS

$1 = 1$
$8 = 1 + 3 + 5$
$27 = 1 + 3 + 5 + 7 + 9$
$64 = 1 + 3 + 5 + 7 + 9 + 11 + 13$
$125 = 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17$

In general: n^3 = sum of first n^2 odd numbers.

EXAMPLES OF PATTERNS

1. Units digit pattern (Squares)
   Possible last digits: 0, 1, 4, 5, 6, 9.
   Not possible: 2, 3, 7, 8.
2. Zeros at the end
   If a number ends with k zeros, its square ends with $2k$ zeros.
3. Parity
   Even^2 = Even, Odd^2 = Odd.
4. Example sets
   $1^2, 9^2, 11^2, 19^2, 21^2, 29^2, \ldots$ end in 1.
   $4^2, 6^2, 16^2, 26^2, \ldots$ end in 6.

FINDING CUBE ROOTS

Methods

1. Prime factorisation: factors in triplets.
2. List cubes in order.
3. Using pattern & estimation.

Example

- $3375 = 3\times3\times3\times5\times5\times5 = (3\times5)^3 = 15^3$
  $\Rightarrow \sqrt[3]{3375} = 15$
- Estimation:
  $15^3 = 3375 < x < 16^3 = 4096 \Rightarrow \sqrt[3]{3500} \approx 15.$

SQUARE & CUBE COMPARISON

Aspect	Square (n^2)	Cube (n^3)
Definition	$n \times n$	$n \times n \times n$
Notation	n^2	n^3
Roots	$\sqrt{n}$	$\sqrt[3]{n}$
Factors (for perfect)	In pairs	In triplets
Unit digit pattern	0,1,4,5,6,9	0,1,8,9 or even 0's
Zeros rule	k zeros → $2k$ zeros	k zeros → $3k$ zeros
Sum of odd numbers	First n odd numbers	First n^2 odd numbers

HISTORICAL CONNECTION

- Babylonians (1700 BCE) compiled lists of squares and cubes for practical use in land measurement and construction.
- In ancient India, squares were called "varga" and cubes were called "ghana".
- Aryabhata (499 CE) defined varga as both the square figure and the square of a number.
- The word mula (root) from Sanskrit (meaning root of a plant, basis, origin) gave rise to the words jidhr (Arabic) and radix (Latin) — from which we get "root" in mathematics.
- Ramanujan (1910s) astonished Hardy by identifying 1729 as the smallest number that is the sum of two cubes in two different ways: $1^3 + 12^3 = 9^3 + 10^3 = 1729$ (Taxicab number).

PUZZLE CORNER

1. Square Pairs!
Arrange numbers 1 to 17 in a row so that the sum of every adjacent pair is a square.
(Hint: unique arrangement exists.)

2. Arrange numbers 1 to 32 in a circle so that adjacent numbers add up to a square.

IMPORTANT FORMULAS

- $n^2 = n \times n$
- $\sqrt{n^2} = n$ (for positive n)
- $n^3 = n \times n \times n$
- $\sqrt[3]{n^3} = n$
- $n^2 - (n-1)^2 = 2n - 1$ (odd number)
- $n^3 - (n-1)^3 = 3n^2 - 3n + 1$
- Sum of first n odd numbers $= n^2$
- Sum of first n^2 odd numbers $= n^3$

KEY TAKEAWAYS

- ✓ Squares and cubes have beautiful patterns.
- ✓ Square numbers are sums of first n odd numbers.
- ✓ Cube numbers are sums of first n^2 odd numbers.
- ✓ Prime factorisation helps identify perfect squares (pairs) and perfect cubes (triplets).
- ✓ Square root (√) and cube root ($\sqrt[3]{y}$) are inverse operations.
- ✓ Patterns help us estimate and solve large problems.

REMEMBER

Look for patterns, factorise smartly, estimate wisely, and numbers will reveal their secrets!

A SQUARE AND A CUBE

WORKSHEET-1

Section A: MCQs (Part-1) (10 × 1 = 10 marks)

1. The total non-square numbers between the squares of 30 and 31 are

(a) 60 (b) 61 (c) 62 (d) 63

2. If a number has 2 or 2 at its unit's place, then its square would end in

(a) 2 (b) 8 (c) 4 (d) 16

3. How many perfect square numbers lie between 16 and 81?

(a) 4 (b) 5 (c) 6 (d) 8

4. Which number is called the Hardy-Ramanujan number?

(a) 1839 (b) 1729 (c) 1949 (d) 1529

5. Cube root of 8 is

(a) 512 (b) 64 (c) 2 (d) 4

6. Which of the following numbers cannot be a perfect square?

(a) 2500 (b) 1600 (c) 961 (d) 468

7. Which of the following numbers would not have the digit 4 at its unit place?

(a) 52^2 (b) 68^2 (c) 34^2 (d) 102^2

8. The partner factors of 25 is/are

(a) 1, 25 (b) 5, 5 (c) 2, 25 (d) Both (a) and (b)

9.A number ending in 9 will have the unit place of its square as:

(a) 1 (b) 3 (c) 9 (d) 6

10.Which of the following is not a unit digit of a square number?

(a) 0 (b) 4 (c) 6 (d) 8

MCQs (Part-2) ($10 \times 1 = 10$ marks)

11.The square root of 0.000025 is:

(a) 0.005 (b) 0.0005 (c) 0.025 (d) 0.00025 2.

12.How many natural numbers lie between 52 and 62?

(a) 9 (b) 10 (c) 11 (d) 12 3.

13. Find the number whose square is 7569:

(a) 87 (b) 86 (c) 88 (d) 89

14.A square board has an area of 144 square units. How long is each side of the board?

(a) 11 units (b) 12 units (c) 13 units (d) 14 units 5.

15.If one member of a Pythagorean triplet is 2m, then the other two members are:

(a) $m, m^2 + 1$ (b) $m^2 + 1, m^2 – 1$ (c) $m^2, m^2 – 1$ (d) $m^2, m + 1$

16. A number has the prime factorisation $\mathbf{2^3 \times 3^2 \times 5^2}$. What is the **smallest number** by which it should be multiplied to make it a perfect square?

(a) 2 (b) 3 (c) 5 (d) 6

17. Which of the following numbers is a **perfect cube but not a perfect square**?

(a) 64 (b) 125 (c) 729 (d) 144

18. The difference between the squares of two consecutive natural numbers is **41**. What is the smaller number?

(a) 19 (b) 20 (c) 21 (d) 22

19. Which of the following numbers has an **integer square root** as well as an **integer cube root**?

(a) 324 (b) 729 (c) 4096 (d) 1000

20. A cube-shaped box has a volume of **2744 cm^3**. What is the length of each edge of the box?

(a) 12 cm (b) 13 cm (c) 14 cm (d) 16 cm

WORKSHEET-2

Section B: Short Answer ($10 \times 2 = 20$ marks)

1. Find the smallest square number which is exactly divisible by 12, 15, and 20.

2. How many non-perfect square numbers lie between the squares of 12 and 13?

3. Is 160 a perfect square? Why or why not?

4. Find the unit digit of the cube of 234.

5. Roxie has 7 dresses, 2 hats, and 3 pairs of shoes. How many different ways can Roxie dress up?

6. Find the smallest number by which 250 must be divided so that the quotient becomes a perfect cube?

7. Find the square root of 2401 by prime factorization method.

8. Find the cube root of 2744, by the prime factorization method.

9. Write 49 as a sum of odd natural numbers.

10. Find the smallest number to multiply by 1800 to make it a perfect square.

Section C: Short Answer (6 × 3 = 18 marks)

11. What would the thickness of a sheet of paper be after 30 folds? Assume that the thickness of the sheet is 0.001 cm.

12. There are 5 bottles in a container. Every day, a new container is brought in. How many bottles would be there after 40 days?

13. . Find the smallest number by which 200 must be multiplied so that the product becomes a perfect cube.

14. Find the smallest number by which 882 must be multiplied to make it a perfect square.

15. Find the square root of 4356, i.e., evaluate $\sqrt{4356}$ by prime factorisation.

16. Find the side of a cubical box whose volume is 1728 cm^3.

Section D: Long Answer (4 × 5 = 20 marks)

17. A school has 1475 students. What is the minimum number of students that need to be added so that the total number of students can be arranged in a perfect square grid?

18. A school has 50 lockers numbered 1 to 50. Students perform the following steps:

- Student 1 opens all lockers.
- Student 2 toggles (opens/closes) every 2nd locker (2,4, 6,…).
- This continues until Student 50 toggles the 50th locker, in the last.

- Which lockers remain open at the end? Explain why.

19. Numbers between the squares of the following pairs: (i) 25 and 26 (ii) 77 and 78

20. By which should we multiply 675 to make it a perfect cube? Also, find the cube root of the perfect cube so obtained.

Case Study Based Questions ($2 \times 4 = 8$ marks)

21. Case Study – **The Stones that Shine ...**

Three daughters with curious eyes, Each got three baskets — a kingly prize. Each basket had three silver keys, Each opens three big rooms with ease. Each room had tables — one, two, three, With three bright necklaces on each, you see. Each necklace had three diamonds so fine…

(i) Can you count these stones that shine?

OR

Can you count the number of tables?

(ii) How many rooms were there altogether?

(iii) Write the product p4 × p6 in exponential form.

22. Math club discussion: During a math club discussion, Arjun told the famous story of Hardy and Ramanujan. Hardy thought 1729 was dull, but Ramanujan replied it was special since $1729 = 1^3 + 12^3 = 9^3 + 10^3$. This made Arjun curious about cubes and their properties. He started exploring other such numbers called "taxicab numbers."

Q1. Find the units digit of the cube of 1729.

Q2. What is special about taxi cab numbers ?

Q3. What is the cube root of 1729?

OR

Can a perfect cube end with the digits 00? Justify your answer.

Self Assessment:

I understood all concepts: ☐ Yes ☐ No

I need more practice: ☐ Yes ☐ No

Name of Concepts having doubts/Not Cleared: ____________________________________

Parent's Assessment:

My ward understood the all the concepts ☐ Yes ☐ No

He/She need more practice ☐ Yes ☐ No

Name of Concepts having doubts/Not Cleared: ____________________________________

Signature of Parent:

Teacher's Assessment:

Answers of Worksheet-1:

Q.No.	1	2	3	4	5	6	7	8	9	10
Ans	a	c	a	b	c	d	a	d	a	d
Q.No.	11	12	13	14	15	16	17	18	19	20
Ans	a	a	a	b	b	a	b	b	c	b

Answers of Worksheet-2:

Q.No	1	2	3	4	5	6	7	8	9	10
Ans	900	24	No	4	42way	3	49	14	49	2
Q.No	11	12	13	14	15	16	17	18	19	20
Ans	10.74km	200 bottles	5	2	66	12cm	46	1,4,9, 16,25, 36,49	(i)50 (ii)154	15
Q.No	21			22						
Ans	2187 OR 243, 81 P^{10}			9, It is smallest number that can be written as the sum oftwo cubes in two different ways. 12.0023 Or no						